Theorem 1.4.1 Let A, B, C, D be sets and $f : A \to B$, $g : B \to C$, $h : C \to D$ be functions such that $\mathrm{Ran}(f) \cap \mathrm{Dom}(g) \neq \varnothing$ and $\mathrm{Ran}(g) \cap \mathrm{Dom}(h) \neq \varnothing$. Then $\mathrm{Id}_B \circ f = f \circ \mathrm{Id}_A = f$ and $(f \circ g) \circ h = f \circ (g \circ h)$.

We refer to the latter property of composition of functions as its "associativity."

Next, we note that if f is a one-to-one function, $f^{-1} \circ f = \mathrm{Id}_{\mathrm{Dom}(f)}$. One can also show that in this case f^{-1} is also one-to-one, $f \circ f^{-1} = \mathrm{Id}_{\mathrm{Ran}(f)}$, and $(f^{-1})^{-1} = f$. The following is an important theorem whose proof follows from the basic properties of bijections.

Theorem 1.4.2 Let A, B, C be sets and $f : A \to B$ and $g : B \to C$ be bijections. Then $f^{-1} : B \to A$ and $g \circ f : A \to C$ are also bijections.

The following corollary provides the basic idea for classifying sets.

Corollary 1.4.1 Let $\mathcal{S}$ be a set of sets and

$$\sim := \{(A, B) \in \mathcal{S}^2 \,|\, \text{There is a bijection } \phi : A \to B\}.$$

Then $\sim$ is an equivalence relation.

Proof: For all $A \in \mathcal{S}$, $\mathrm{Id}_A : A \to A$ is a bijection. Therefore, $A \sim A$. This shows that $\sim$ is reflexive. For all $A, B \in \mathcal{S}$, if $A \sim B$, there is a bijection $\phi : A \to B$. But then $\phi^{-1} : B \to A$ is also a bijection, $B \sim A$, and $\sim$ is symmetric. Finally, let $A, B, C \in \mathcal{S}$ be such that $A \sim B$ and $B \sim C$. Then there are bijections $\phi : A \to B$ and $\psi : B \to C$. Because ϕ and ψ are bijection, the function $\psi \circ \phi : A \to C$ is also a bijection. This shows that $A \sim C$. Hence $\sim$ is transitive.

In view of this corollary, *two sets A and B are said to be **equivalent** if there is a bijection $\phi : A \to B$, i.e., $A \sim B$. The equivalence classes of $\sim$ are called **cardinal numbers***. We use cardinal number of sets to classify them. One can for example show that $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are equivalent sets (they have the same cardinal number), but that $\mathbb{R}$ is not equivalent to $\mathbb{N}$ (the cardinal number of $\mathbb{R}$ is different from that of $\mathbb{N}$.) For a further discussion of cardinal numbers, we refer the reader to texts on Set Theory, e.g., [1].

Problems

Problem 1.1 Let a be a statement. Show that $a \Rightarrow \neg a$ is not a contradiction.

Problem 1.2 Let c and d be statements and $h := ((c \Rightarrow d) \wedge (d \Rightarrow c))$. Show that if h is false, then so is $c \Leftrightarrow d$.

Problem 1.3 Prove the identities (1.2) – (1.4).

Problem 1.4 Let for all $a \in \mathbb{R}^+$, A_a be the open interval $]-a, a[$. Show that $\bigcup_{a\in\mathbb{R}^+} A_a = \mathbb{R}$ and $\bigcap_{a\in\mathbb{R}^+} A_a = \{0\}$.

Problem 1.5 Prove Theorem 1.3.1.

Problem 1.6 Prove Theorem 1.3.2.

Problem 1.7 Show that for every $a, b \in \mathbb{R}$ the relation f defined in (1.24) is an everywhere-defined function that is both one-to-one and onto provided that $a \neq 0$.

Problem 1.8 Prove Theorem 1.4.1.

Problem 1.9 Let $f : A \to B$ be a one-to-one function. Show that f^{-1} is also one-to-one and we have $f \circ f^{-1} = \mathrm{Id}_{\mathrm{Ran}(f)}$ and $(f^{-1})^{-1} = f$.

Problem 1.10 Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions defined by $f(x) := 1 + \sqrt{-x}$ and $g(x) := 1/\sqrt{1 - x^2}$. Determine the domain and range of f, g, $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$.

Problem 1.11 Prove Theorem 1.4.2.

Problem 1.12 Let $n \in \mathbb{Z}^+$, $I_n := \{1, 2, \cdots, n\}$, $a_1, a_2, \cdots, a_n$ be distinct real numbers, and $A := \{a_1, a_2, \cdots, a_n\}$. Show that there is a bijection $\phi : A \to I_n$.

Problem 1.13 Construct a bijection $\psi : \mathbb{N} \to \mathbb{Z}^+$.

Problem 1.14 Construct a bijection $\chi : \mathbb{Z} \to \mathbb{N}$.